Some Common Fixed Point Theorems for a Pair of Non expansive Mappings in Generalized Exponential Convex Metric Space

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Abstract: In this paper, we considered an Ishikawa iteration schema with errors to approximate the unique common fixed point for a Pair of Non expansive Mappings in Generalized Convex Metric Space.

Key words: Common fixed point, Non expansive mapping, and convex metric space.

1. Introduction

In the literature common fixed point theory many authors proved several convergence theorems for common fixed points of mean non expansive mappings. Also studied the same problem; (Gu and Li , 2008) they considered the Ishikawa iteration process to approximate the common fixed point of mean non expansive mappings in uniformly convex Banach space. (Takahashi ,1970) first introduced a notion of convex metric space, which is more general space, and each linear normed space is a special example of the space. Late on, (Ciric ,2003) proved the convergence of an Ishikawa type iteration process to approximate the common fixed point of a pair of mappings under condition B. Very recently, (Wang and Liu, 2009) give some sufficiency and necessary conditions for an Ishikawa type iteration process with errors to approximate a common fixed point of two mappings in generalized convex metric space. Inspired and motivated by the above facts, we will consider the Ishikawa type iteration process with errors, which converges to the unique common fixed point of the pair of asymptotically nonexpansive mappings in generalized convex metric space. Our results extend and improve the corresponding results.

2. Definitions and Preliminaries

Definition 2.1 (see Takahashi ,1970 ). Let \((X,d)\) be a metric space, and \(I = [0,1] \). A mapping \(\omega: X^2 \times I \rightarrow X\) is said to be exp-convex structure on \(X\), if for any \((x,y,\alpha)\in X\) and \(u \in X\), the following inequality holds:
\[
e^{d(a(x,y,\alpha),u)} \leq \alpha e^{d(x,u)} + (1-\alpha)e^{d(y,u)}  \tag{2.1}\]

If \((X,d)\) is a metric space with a convex structure \(\omega\), then \((X,d)\) is called a convex metric space. Moreover, a nonempty subset \(E\) of \(X\) is said to be convex if \((x,y,\alpha)\in X\) for all \((x,y,\alpha)\in E^2 \times I\).

Definition 2.2 (see Y.-X. Tian, 2005). Let \((X,d)\) be a metric space, and \(I = [0,1]\) and \(\{a_n\}, \{b_n\}, \{c_n\}\) real sequences in \([0,1]\) with \(a_n + b_n + c_n = 1\). A mapping \(\omega: X^3 \times I^3 \rightarrow X\) is said to be convex structure on \(X\), if for any \((x,y,z,a_n,b_n,c_n)\in X^3 \times I^3\) and \(u \in X\), the following inequality holds:
\[
d(\omega(x,y,z,a_n,b_n,c_n)) \leq a_n d(x,u) + b_n d(y,u) + c_n d(z,u) \tag{2.2}\]
If \((X,d)\) is a metric space with a convex structure \(\omega\), then \((X,d)\) is called a generalized convex metric space. Moreover, a nonempty subset \(E\) of \(X\) is said to be convex if \(\omega(x,y,z,a_n,b_n,c_n)\in E\), for all \((x,y,z,a_n,b_n,c_n)\in E^3 \times I^3\).

**Remark 2.3.** It is easy to see that every generalized convex metric space is a convex metric space (let \(c_n = 0\)).

**Definition 2.4.** Let \((X,d)\) be a metric space, and \(I = [0,1]\) and \(\{a_n\},\{b_n\},\{c_n\}\) real sequences in \([0,1]\) with \(a_n + b_n + c_n = 1\). A mapping \(\omega: X^3 \times I^3 \rightarrow X\) is said to be convex structure on \(X\), if for any \((x,y,z,a_n,b_n,c_n)\in X^3 \times I^3\) and \(u \in X\), the following inequality holds:

\[
d(x,y,z) \leq \log(a_n e^{d(u,x)} + b_n e^{d(y,x)} + c_n e^{d(z,x)})
\]

**Definition 2.5.** Let \((X,d)\) be a generalized convex metric space with a convex structure \(\omega: X^3 \times I^3 \rightarrow X\) and \(E\) a nonempty closed convex subset of \(X\). Let \(S,T:E \rightarrow E\) be a pair of asymptotically nonexpansive mappings, and \(\{a_n\},\{b_n\},\{c_n\}\) six sequences in \([0,1]\) satisfying \(a_n + b_n + c_n = 1\), \(n = 1,2,\ldots\) for any given \(x_n \in E\), define a sequence \(\{x_n\}\) as follows:

\[
x_{n+1} = \omega(x_n,S^n y_n,u_n,a_n,b_n,c_n)
\]

**Remark 2.6** (see Wang and Liu, 2009). Let \(E\) be a nonempty closed convex subset of complete convex metric space \(X\) and \(S,T:E \rightarrow E\) uniformly quasi-Lipschitzian mappings with \(L > 0\) and \(L > 0\) and \(F = F(S) \cap F(T) \neq \emptyset\) \((F(T) = \{x \in X : Tx = x\})\).

Suppose that \(\{x_n\}\) is the Ishikawa type iteration process with errors defined by \([2.1]\), \(\{u_n\}\), \(\{v_n\}\) satisfy (A), and \(\{a_n\},\{b_n\},\{c_n\}\) \(\{a_n',\{b_n',\{c_n'\}\} six sequences in [0,1] satisfying\)

\[
a_n + b_n + c_n = a_n' + b_n' + c_n' = 1, \sum_{n=0}^{\infty} (b_n + c_n) < \infty, \quad 2.6
\]

then \(\{x_n\}\) converges to a fixed point of \(S\) and \(T\) if and only if \(\lim \inf_{n \to \infty} d(x_n,F) = 0\), where \(d(x,F) = \inf \{d(x,p) : p \in F\}\).

**Remark 2.7.** Let \(F(T) = \{x \in X : Tx = x\} \neq \emptyset\). A mapping \(T:X \rightarrow X\) is called uniformly quasi-Lipschitzian if there exists \(L > 0\) such that

\[
d(T^n x,p) \leq Ld(x,p) \quad 2.7
\]

for all \(x \in X, p \in F(T), n \geq 1\). Reason behind this study is the elementary inequality

\[
\sqrt{ab} \leq \frac{a + b}{2} (a > 0, b > 0)
\]

Asserts the convexity of the function \(e^x\), of which may easily convince oneself by making the substitution
\[ a = e^x \quad \text{and} \quad b = e^y \quad \text{which gives} \quad e^{x+y} \leq e^x + e^y \quad \text{--- (B)} \]

The study of the conditions under which the superpositions of the functions of certain classes turn out to be convex or concave is of definite interest. Let \( X \) be a nonempty convex subset of \( R^n \), show that \( \text{set} \ \{ x, y \} \) is convex for \( \forall \ x, y \in X \). We assumed that \( X \) is a nonempty convex subset of \( R^n \), such that \( \text{set} \ \{ x, y \} \) is convex for an arbitrary \( y \in X \).

The relation \( x - y + y \in X \) is satisfied. Let us put \( y = x + t \). Hence for any real number \( \alpha \in (0,1) \), we have

\[
\forall \ x \in X \quad \alpha e^{(x+y)} \in \alpha e^X \land (1-\alpha)e^y \in (1-\alpha)e^X,
\]

And so

\[
\forall \ x \in X \quad \alpha e^{(x+y)} + (1-\alpha)e^y \in \alpha e^X + (1-\alpha)e^X.
\]

By assumption, the set \( e^X \) is convex.

We obtained \( \alpha e^X + (1-\alpha)e^y = e^X \).

\[
\forall \ x, y \in X \quad \log (\alpha e^{(x+y)} + (1-\alpha)e^y) \in X, \quad \text{Holds for any} \ \alpha \in (0,1) \ \text{and any arbitrary point} \ y \in X. \ \text{This means by definition, that} \ X \ \text{is convex.}
\]

3. Main Results

Now, we will prove the strong convergence of the iteration scheme (2.4) to the unique common fixed point of a pair of asymptotically nonexpansive mappings \( S \) and \( T \) in complete generalized convex metric spaces.

**Theorem 3.1.** Let \( E \) be a nonempty closed convex subset of complete generalized exp-convex metric space \( X \), and \( S, T : E \to E \) a pair of asymptotically nonexpansive mappings with \( b \neq 0 \) and \( F = F(S) \cap F(T) \neq \emptyset \). Suppose \( \{ x_n \} \) as in (2.4), \( \{ u_n \} \), \( \{ v_n \} \) satisfy (B), and \( \{ a_n \}, \{ b_n \}, \{ c_n \}, \{ a'_n \}, \{ b'_n \}, \{ c'_n \} \) are six sequences in \([0,1]\) satisfying

\[
a_n + b_n + c_n = a_n' + b_n' + c_n' = 1, \quad \sum_{n=0}^{\infty} (b_n + c_n) < \infty,
\]

then \( \{ x_n \} \) converge to the unique common fixed point of \( S \) and \( T \) if and only if

\[
\lim \inf_{n \to \infty} d(x_n, F) = 0,
\]

where

\[
d(x, F) = \inf \{ d(p, x) : p \in F \}.
\]

**Proof.** The necessity of conditions is obvious. Thus, we will only prove the sufficiency. Let \( p \in F \), for all \( x \in E \),

\[
e^{d(x,p)} \leq \log \left( a e^{d(x,p)} + b \left[ e^{d(x,Sx)} + e^{d(p,p)} \right] + c \left[ e^{d(x,p)} + e^{d(p,Sx)} \right] \right)
\]

\[
\leq \log \left( a e^{d(x,p)} + b \left[ e^{d(x,p)} + e^{d(p,Sx)} \right] + c \left[ e^{d(x,p)} + e^{d(p,Sx)} \right] \right)
\]

implies

\[
(1 - b - c) e^{d(x,p)} \leq (a + b + c) e^{d(x,p)}
\]

which yield (using the fact that \( a + 2b + 2c \leq 1 \) and \( b \neq 0 \))

\[
e^{d(Sx,p)} \leq Ke^{d(x,p)},
\]

where \( 0 < K = (a + b + c) / (1 - b - c) \leq 1 \). Similarly, we also have \( e^{d(Tx,p)} \leq Ke^{d(x,p)} \).

By Remark 2.7, we get that \( S \) and \( T \) are two uniformly quasi-Lipschitzian mappings (with \( L = L' = K > 0 \)). Therefore, from Theorem 1.6, we know that \( \{ x_n \} \) converges to a common fixed point of \( S \) and \( T \).

Finally, we prove the uniqueness. Let \( p_1 = S p_1 = T p_1, p_2 = S p_2 = T p_2 \), then by (*), we have

\[
d(p_1, p_2) \leq \log \left( a e^{d(p_1,p_2)} + b \left[ e^{d(p_1,p_1)} + e^{d(p_2,p_2)} \right] + c \left[ e^{d(p_1,p_2)} + e^{d(p_2,p_1)} \right] \right)
\]

\[
\leq \log ((a + 2c) e^{d(p_1,p_1)}).
\]

\[
(a + 2c) = 1
\]

Since \( (a + 2c) < 1 \), we obtain \( p_1 = p_2 \). This completes the proof.
Remark 3.2. (i) We consider a sufficient and necessary condition for the Ishikawa type iteration process with errors in complete generalized convex metric space; our mappings are the more general mappings (a pair of asymptotically nonexpansive mappings), so our result extend and generalize the corresponding results.

(ii) Since $\{x_n\}$ converges to the unique fixed point of $S$ and $T$, we have improved Theorem 1.6 in (C. Wang and L. W. Liu 2009).

Corollary 3.3. Let $E$ be a nonempty closed convex subset of Banach space $X$, $S, T : X \to X$ a pair of asymptotically nonexpansive mappings, that is,

$$e^{d(x,y)} \leq \log\left(a e^{d(x,y)} + b \left[e^{d(Sx,y)} + e^{d(Ty,y)}\right] + c\left[e^{d(x,y)} + e^{d(y,Sx)}\right]\right)$$

with $b \neq 0$, and $F = F(S) \cap F(T) \neq \emptyset$. For any given $x_1 \in E$, $\{x_n\}$ is an Ishikawa type iteration process with errors defined by

$$x_{n+1} = a_n x_n + b_n S^ny_n + c_n u_n$$

$$y_n = a'_n x_n + b'_n T^ny_n + c'_n v_n$$

where $\{u_n\}$, $\{v_n\} \in E$ are two bounded sequences

and $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are six sequences in $[0,1]$ satisfying

$$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, \sum_{n=0}^{\infty} (b_n + c_n) < \infty.$$  

Then, $\{x_n\}$ converges to the unique common fixed point of $S$ and $T$ if and only if

$$\liminf_{n \to \infty} e^{d(x_n,p)} = 0,$$

where

$$e^{d(x,y)} = \inf\left\{e^{d(x,p)} : p \in F\right\}.$$  

Proof. From the proof of Theorem 3.1, we have

$$e^{d(x,y)} \leq K e^{d(x,p)}, e^{d(y,z)} \leq K e^{d(x,p)}$$

where $K = (a + b + c)/(1 - b - c)$. Hence, $S$ and $T$ are two uniformly nonexpansive mappings in Banach space. Since Theorem 1.6 also holds in Banach spaces, we can prove that there exists a $p \in F$ such that

$$\lim_{n \to \infty} ||x_n - p|| = 0.$$ . The proof of uniqueness is the same to that of Theorem 2.1. Therefore, $\{x_n\}$ converges to the unique common fixed point of $S$ and $T$.

Corollary 3.4. Let $E$ be a nonempty closed convex subset of Banach space $X$, $S, T : X \to X$ a pair of asymptotically nonexpansive mappings, that is,

$$e^{d(x,y)} \leq \log\left(a e^{d(x,y)} + b \left[e^{d(x,Sx)} + e^{d(y,Ty)}\right] + c\left[e^{d(x,y)} + e^{d(y,Sx)}\right]\right)$$

with $b \neq 0$, and $F = F(S) \cap F(T) \neq \emptyset$. For any given $x_1 \in E$, $\{x_n\}$ is an Ishikawa type iteration process with errors defined by

$$x_{n+1} = \log\left(\alpha_n e^{x_n} + (1 - \alpha_n) e^{y_n}\right)$$

$$y_n = \log\left(\beta_n e^{x_n} + (1 - \beta_n) e^{y_n}\right)$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0,1]$ satisfying

$$\sum_{n=1}^{\infty} (1 - \alpha_n) < \infty.$$ . Then, $\{x_n\}$ converges to the unique common fixed point of $S$ and $T$ if and only if

$$\liminf_{n \to \infty} e^{d(x_n,p)} = 0,$$

where

$$e^{d(x,y)} = \inf\left\{e^{d(x,p)} : p \in F\right\}.$$  

Proof. Let $a_n = \alpha_n$ and $a'_n = \beta_n$ and $c_n = c'_n$. The result can be deduced immediately from Corollary 3.3. This completes the proof.

Conclusion: As we know various applications of fractal sets in cryptography, stegnography digital signature, commitment scheme, key agreement protocol, that means all communications between two communicators with security. The application of
Ishikawa iterations in fractal sets is well known (Yashwant S Chauhan, Rajeshri Rana, Ashish Negi, 2010). Our approach provides more generalized convex structure on the basis of equation 2.8, it extends the results of previous literature. Our approach will lead the platform for extension of convex structure to invex structure in common fixed point literature for future endeavors.

References


